

The c -function for non-compactly causal symmetric spaces

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Introduction

In this paper we prove a product formula for the c -function associated to a non-compactly causal symmetric space \mathcal{M} . Let us recall here the basic facts. Let G be a connected semisimple Lie group, $\tau : G \rightarrow G$ be a non-trivial involution and $H = G^\tau$. Then $\mathcal{M} := G/H$ is a semisimple symmetric space. The space \mathcal{M} is called non-compactly causal, if $\mathfrak{q} := \{X \in \mathfrak{g} : \tau(X) = -X\}$ contains an open H -invariant hyperbolic cone $C \neq \emptyset$. In this case $S := H \exp(C)$ is a open subsemigroup of G . A *spherical function* on \mathcal{M} is an H -biinvariant continuous function on $S/H \subseteq \mathcal{M}$, which defines an eigendistribution of the algebra of H -invariant differential operators on \mathcal{M} , see [FHÓ94], [KNÓ98], [Ól97]. There exists a maximal abelian hyperbolic subspace \mathfrak{a} of \mathfrak{q} such that $C = \text{Ad}(H).(\mathfrak{a} \cap C)$. Let

$$\varphi_\lambda(g.x_0) = \int_H a_H(gh)^{\lambda-\rho} d\mu_H(h)$$

be a spherical function given by a convergent integral similar to the expression for the spherical functions on the Riemannian symmetric spaces G/K . Here $x_0 \in \mathcal{M}$ is the coset $\{H\}$ and $a_H(g) \in A := \exp(\mathfrak{a})$ is determined by $g \in Ha_H(g)N$. The asymptotic behaviour of $\varphi_\lambda(a.x_0)$ along $S \cap A$ is given by $\varphi_\lambda(a.x_0) \sim c(\lambda)a^{\lambda-\rho}$, where ρ is half the sum over the positive roots counted with multiplicities. The function $c(\lambda)$ is the *c-function of the space \mathcal{M}* . It turns out that the c -function is a product of two c -functions, $c(\lambda) = c_\Omega(\lambda)c_0(\lambda)$ where $c_0(\lambda)$ is the Harish-Chandra c -function of a Riemannian subsymmetric space $G(0)/K(0)$ and $c_\Omega(\lambda)$ is a function associated to the real bounded symmetric domain $H/(H \cap K)$, where K is a τ -stable maximal compact subgroup of G . The c -function was first introduced by Oshima-Sekiguchi in [OS80], whereas $c_\Omega(\lambda)$ was first introduced in [FHÓ94].

The c -function for a Riemannian symmetric space G/K can be written as a product of c -functions of rank one symmetric spaces associated to each restricted root of \mathfrak{g} (Gindikin-Karpelevic formula). For general non-Riemannian symmetric spaces G/H one cannot expect this type of result. However, for non-compactly causal symmetric spaces we show in this paper (cf. Theorem III.5) that such a product formula holds. The case of Cayley type spaces has already been treated by J. Faraut in [Fa95] by the use of Jordan algebra methods and in [Gr97] the case $\text{Sl}(n, \mathbb{R})/\text{SO}(p, q)$ is dealed. The approach presented here is general, different and relies on new insights on the fine convex geometry of the real bounded symmetric domain Ω (cf. Theorem II.5 and Theorem II.7.)

Our result has important applications. The c -function was the last unknown part in the formula for the formal degree of the spherical holomorphic discrete series representations

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representations (cf. [Kr99]). Further it gives us important information on the normalized spherical functions $\tilde{\varphi}_\lambda := c_\Omega(\lambda)^{-1} \varphi_\lambda$. One knows that the function $\lambda \mapsto \tilde{\varphi}_\lambda(s.x_0)$ has a meromorphic continuation to $\mathfrak{a}_\mathbb{C}^*$ (cf. [Ól97]) and the product formula gives us important information on the poles. In particular, this allows more detailed analysis of the spherical Laplace transform, in particular Paley-Wiener type theorems.

I. Non-compactly causal symmetric spaces and Lie algebras

In this section we introduce notation and recall some facts concerning non-compactly causal symmetric Lie algebras and their associated symmetric spaces. Our source of reference is [HiÓl96].

Algebraic preliminaries

Let \mathfrak{g} be a simple finite dimensional real Lie algebra. Let $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ be a non-trivial involution. Then (\mathfrak{g}, τ) is a *symmetric Lie algebra*. We write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the τ -eigenspace decomposition of \mathfrak{g} corresponding to the eigenvalues $+1$ and -1 . Let θ be a Cartan involution of \mathfrak{g} which commutes with τ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition.

For $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ let $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{b}) := \{X \in \mathfrak{a} : [X, Y] = 0, Y \in \mathfrak{b}\}$ be the *centralizer of \mathfrak{b} in \mathfrak{a}* . We call (\mathfrak{g}, τ) *non-compactly causal*, or simply NCC, if $\mathfrak{z}_{\mathfrak{q} \cap \mathfrak{p}}(\mathfrak{h} \cap \mathfrak{k}) \neq \{0\}$. We call (\mathfrak{g}, τ) *non-compactly Riemannian (NCR)* if τ is a Cartan involution. If not otherwise stated from now on (\mathfrak{g}, τ) denotes a NCC symmetric Lie algebra. Then $\mathfrak{z}_{\mathfrak{q} \cap \mathfrak{p}}(\mathfrak{h} \cap \mathfrak{k}) = \mathfrak{z}(\mathfrak{q} \cap \mathfrak{p}) = \mathbb{R}X_0$ is one dimensional. Let $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$ be a maximal abelian subspace and note that $\mathbb{R}X_0 \subseteq \mathfrak{a}$ and that \mathfrak{a} is maximal abelian in \mathfrak{p} . We write $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ for the root system of \mathfrak{g} with respect to \mathfrak{a} and

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

for the corresponding root space decomposition. We write $\mathfrak{g}(0) := \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$ and note that $(\mathfrak{g}(0), \tau(0))$, with $\tau(0) := \tau|_{\mathfrak{g}(0)}$, is NCR. If $\alpha \in \Delta$ then either $\mathfrak{g}^\alpha \subseteq \mathfrak{g}(0)$ or $\mathfrak{g}^\alpha \subseteq \mathfrak{q} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p}$. A root $\alpha \in \Delta$ is called *compact* if $\mathfrak{g}^\alpha \subseteq \mathfrak{g}(0)$ and *non-compact* if $\mathfrak{g}^\alpha \subseteq \mathfrak{q} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p}$. We write Δ_k and Δ_n for the collection of compact and non-compact roots, respectively. Note that $\Delta = \Delta_k \dot{\cup} \Delta_n$.

We can and will normalize X_0 such that $\text{Spec}(\text{ad } X_0) = \{-1, 0, 1\}$. Then $\Delta_k = \{\alpha \in \Delta : \alpha(X_0) = 0\}$ and we can choose a positive system Δ^+ of Δ such that

$$\Delta_n^+ := \Delta_n \cap \Delta^+ = \{\alpha \in \Delta_n : \alpha(X_0) = 1\}$$

and such that $\Delta_k^+ := \Delta_k \cap \Delta^+$ is a positive system in Δ_k . Let $\Delta^- := -\Delta^+$, $\Delta_n^- := -\Delta_n^+$ and $\Delta_k^- := -\Delta_k^+$.

We recall now few facts about the structure of the root system Δ . Two roots $\alpha, \beta \in \Delta$ are said to be *strongly orthogonal* if $\alpha \pm \beta$ is not a root. Let $\Gamma := \{\gamma_1, \dots, \gamma_r\}$ be a system of strongly orthogonal roots in Δ_n^+ of maximal length, i.e., Γ consists of pairwise strongly orthogonal roots and has maximal number of elements with respect to this property. We set

$$\mathcal{W} := N_{\text{Inn}(\mathfrak{h} \cap \mathfrak{k})}(\mathfrak{a}) / Z_{\text{Inn}(\mathfrak{h} \cap \mathfrak{k})}(\mathfrak{a})$$

and call \mathcal{W} the *small Weyl group* of Δ .

Proposition I.1. *For the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ of a non-compactly causal symmetric Lie algebra (\mathfrak{g}, τ) the following assertions hold:*

- (i) *The root system Δ is reduced, i.e., if $\alpha \in \Delta$ then $2\alpha \notin \Delta$. In particular, there exists at most two root lengths.*
- (ii) *All long roots in Δ_n^+ are conjugate under the small Weyl group \mathcal{W} . Moreover, all roots γ_i , $1 \leq i \leq r$, are long.*
- (iii) *Write $\Delta_{n,s}^+$ for the short roots in Δ_n^+ . Then, if $\Delta_{n,s}^+ \neq \emptyset$, one has*

$$\Delta_{n,s}^+ = \left\{ \frac{1}{2}(\gamma_i + \gamma_j) : 1 \leq i < j \leq r \right\}$$

and all elements of $\Delta_{n,s}^+$ are conjugate under \mathcal{W} .

Proof. (i) [HiÓl96, Th. 3.2.4] or [NÓ99, Lemma 2.12].

(ii) [NÓ99, Lemma 2.26].

(iii) [NÓ99, Lemma 2.22, Lemma 2.24]. ■

For $\alpha \in \Delta$ let $H_\alpha \in \{[X, \tau(X)] : X \in \mathfrak{g}^\alpha\} \subseteq \mathfrak{a}$ be such that $\alpha(H_\alpha) = 2$. For each $1 \leq i \leq r$ let $H_i = H_{\gamma_i}$. We set $\mathfrak{c} := \text{span}_{\mathbb{R}}\{H_1, \dots, H_r\} \subseteq \mathfrak{a}$ and write \mathfrak{b} for the orthogonal complement of \mathfrak{c} in \mathfrak{a} with respect to the Cartan-Killing form; in particular $\mathfrak{a} = \mathfrak{c} \oplus \mathfrak{b}$.

Proposition I.2. *The positive system Δ_k^+ can be chosen such that for the restriction of $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ to \mathfrak{c} the following assertions hold:*

$$\Delta_n^+|_{\mathfrak{c}} = \left\{ \frac{1}{2}(\gamma_i + \gamma_j) : 1 \leq i, j \leq r \right\} \cup \left\{ \frac{1}{2}\gamma_i : 1 \leq i \leq r \right\},$$

$$\Delta_k^+|_{\mathfrak{c}} \setminus \{0\} = \left\{ \frac{1}{2}(\gamma_i - \gamma_j) : 1 \leq j < i \leq r \right\} \cup \left\{ -\frac{1}{2}\gamma_i : 1 \leq i \leq r \right\}.$$

Moreover, the second sets in the two unions from above may or may not occur simultaneously.

Proof. [NÓ99, Th. 2.21] or [Kr99, Th. IV.4]. ■

Since we have free choice for Δ_k^+ we assume in the sequel that $\Delta_k^+|_{\mathfrak{c}} \setminus \{0\} \subseteq \left\{ \frac{1}{2}(\gamma_i - \gamma_j) : 1 \leq j < i \leq r \right\} \cup \left\{ -\frac{1}{2}\gamma_i : 1 \leq i \leq r \right\}$.

Lemma I.3. *Assume that $\Delta_{n,s} \neq \emptyset$ and let Π_k be the set of simple roots corresponding to Δ_k^+ . Then there exists $\beta_1, \dots, \beta_m \in \mathfrak{b}^*$, $\beta_j(X_0) = 0$, and $\delta_1, \dots, \delta_l \in \mathfrak{b}^*$, $\delta_i(X_0) = \frac{1}{2}$, such that*

$$\Pi_k = \left\{ \frac{1}{2}(\gamma_{i+1} - \gamma_i) : 1 \leq i \leq r-1 \right\} \cup \{\beta_1, \dots, \beta_m\} \cup \left\{ -\frac{1}{2}\gamma_r + \delta_i : 1 \leq i \leq l \right\}.$$

Here the last set occurs if and only if there exists half roots in $\Delta|_{\mathfrak{c}}$.

Proof. For each $\alpha \in \Delta$ let s_α denote the corresponding reflection. Then $s_{\gamma_j}(\frac{1}{2}(\gamma_i + \gamma_j)) = \frac{1}{2}(\gamma_i - \gamma_j)$, $i \neq j$ together with Proposition I.1(iii) shows that $\Delta_k \supseteq \left\{ \frac{1}{2}(\gamma_i - \gamma_j) : 1 \leq i \neq j \leq r \right\}$. Thus Proposition I.2 yields that

$$\begin{aligned} \Delta_k^+ &\subseteq \left\{ \frac{1}{2}(\gamma_i - \gamma_j) : 1 \leq j < i \leq r \right\} + (\mathfrak{b}^* \cap X_0^\perp) \\ &\quad \cup \left\{ -\frac{1}{2}\gamma_i : 1 \leq i \leq r \right\} + \left\{ \delta \in \mathfrak{b}^* : \delta(X_0) = \frac{1}{2} \right\} \cup \mathfrak{b}^*. \end{aligned}$$

Now the assertion follows easily from Proposition I.2 and the fact that Δ is a root system. ■

We define the *maximal cone* in \mathfrak{a} is defined by

$$C_{\max} := \{X \in \mathfrak{a}: (\forall \alpha \in \Delta_n^+) \alpha(X) \geq 0\}.$$

Lemma I.4. *Let $X_0 = X_0^b + X_0^c$ with $X_0^b \in \mathfrak{b}$ and $X_0^c \in \mathfrak{c}$. Then we have $X_0^b, X_0^c \in C_{\max}$.*

Proof. First note that $X_0^c = \frac{1}{2}(H_1 + \dots + H_r)$ and so $X_0^c \in C_{\max}$ by Proposition I.2. To show $X_0^b \in C_{\max}$ let $\alpha \in \Delta_n^+$. Then Proposition I.2 shows that $\alpha = \frac{1}{2}(\gamma_i + \gamma_j) + \beta$ with $\beta \in \mathfrak{b}^*$, $\beta(X_0) = \beta(X_0^b) = 0$, or $\alpha = -\frac{1}{2}\gamma_i + \delta$ with $\delta \in \mathfrak{b}^*$ and $\delta(X_0) = \delta(X_0^b) = \frac{1}{2}$. In any case we have $\alpha(X_0^b) \geq 0$ concluding the proof of the lemma. ■

Finally we define subalgebras of \mathfrak{g} by

$$\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \quad \overline{\mathfrak{n}} := \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^\alpha, \quad \mathfrak{n}_k^\pm := \bigoplus_{\alpha \in \Delta_k^\pm} \mathfrak{g}^\alpha, \quad \mathfrak{n}_n^\pm := \bigoplus_{\alpha \in \Delta_n^\pm} \mathfrak{g}^\alpha$$

and note that $\mathfrak{n} = \mathfrak{n}_n^+ \rtimes \mathfrak{n}_k^+$ and $\overline{\mathfrak{n}} = \mathfrak{n}_n^- \rtimes \mathfrak{n}_k^-$ are semidirect products.

Analytic preliminaries

Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let G be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to \mathfrak{g} . Let $H = G^\tau = \{X \in G: \tau(g) = g\}$. We write A , K , N , \overline{N} , N_k^\pm , N_n^\pm for the analytic subgroups of G which correspond to \mathfrak{a} , $\mathfrak{g}(0)$, \mathfrak{h} , \mathfrak{k} , \mathfrak{n} , $\overline{\mathfrak{n}}$, \mathfrak{n}_k^\pm , \mathfrak{n}_n^\pm . Note that the groups A , N , \overline{N} , N_k^\pm , N_n^\pm are all simply connected and that the corresponding exponential mappings $\exp_A: \mathfrak{a} \rightarrow A$, $\exp_N: \mathfrak{n} \rightarrow N$ etc. are all diffeomorphisms. Let $G(0) = Z_G(X_0) = \{g \in G: \text{Ad}(g).X_0 = X_0\}$. Then H and $G(0)$ are τ and θ invariant, $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$ and $G(0) = (H \cap K)\exp(\mathfrak{q} \cap \mathfrak{p})$.

The Lie algebra \mathfrak{g} decomposes as $\mathfrak{g} = \mathfrak{h} + \mathfrak{a} + \mathfrak{n}$ and the multiplication mapping

$$H \times A \times N \rightarrow G, \quad (h, a, n) \mapsto han$$

is an analytic diffeomorphism onto its open image HAN .

Note that $\overline{N} = N_n^- \rtimes N_k^-$. We have

$$(1.1) \quad \overline{N} \cap HAN = \exp(\Omega)N_k^- = N_k^- \exp(\Omega)$$

with $\Omega \cong H/H \cap K$ a real bounded symmetric domain in \mathfrak{n}_n^- .

II. The geometry of the real bounded symmetric domain Ω

We denote by κ the Cartan-Killing form on \mathfrak{g} and define an inner product on \mathfrak{g} by $\langle X, Y \rangle := -\kappa(X, \theta(Y))$ for $X, Y \in \mathfrak{g}$. Let $X_i \in \mathfrak{g}^{\gamma_i}$ be such that $H_i = [X_i, X_{-i}]$, with $X_{-i} = \tau(X_i)$. By [HiOl96] and Herman's Convexity Theorem we have

$$(2.1) \quad \Omega = \{X \in \mathfrak{n}_n^-: \|\text{ad}(X + \tau(X))\| < 1\}$$

$$(2.2) \quad = \text{Ad}(H \cap K) \cdot \left\{ \sum_{j=1}^r t_j X_{-j} : -1 < t_j < 1, j = 1, \dots, r \right\},$$

where $\|\cdot\|$ denotes the operator norm corresponding to the scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . Note that (2.1) implies that Ω is a convex balanced subset of \mathfrak{n}_n^- .

Remark II.1. Recall the definition of the maximal cone C_{rmmax} in \mathfrak{a} . Then it is clear from the characterization (2.1) of Ω that $e^{\text{ad}X}\cdot\Omega \subseteq \Omega$ for all $X \in C_{\max}$. We also have a *minimal cone* in \mathfrak{a} defined by

$$C_{\min} := \text{cone}(\{[X, \tau(X)]: X \in \mathfrak{g}^\alpha, \alpha \in \Delta^+\}) = \overline{\sum_{\alpha \in \Delta_n^+} \mathbb{R}^+ H_\alpha}.$$

We note that $C_{\min} \subseteq C_{\max}$ and in particular $H_i \in C_{\max}$ for each $1 \leq i \leq r$. \blacksquare

The following concept turns out to be very useful for the investigation of the fine convex geometry of Ω .

Definition II.2. (Oshima-Sekiguchi) By a *signature* of Δ we understand a map $\varepsilon: \Delta \rightarrow \{-1, 1\}$ with the following properties:

- (S1) $\varepsilon(\alpha) = \varepsilon(-\alpha)$ for all $\alpha \in \Delta$.
- (S2) $\varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta)$ for all $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$. \blacksquare

If $\varepsilon: \Delta \rightarrow \{-1, 1\}$ is a signature then $\theta_\varepsilon: \mathfrak{g} \rightarrow \mathfrak{g}$ defined by $\theta_\varepsilon(X) = \varepsilon(\alpha)\theta(X)$, $X \in \mathfrak{g}^\alpha$ and $\theta_\varepsilon|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})} = \theta|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})}$ is an involution on \mathfrak{g} that commutes with θ (see [OS80, Def. 1.2]). As $\tau|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})} = \theta|_{\mathfrak{z}_\mathfrak{g}(\mathfrak{a})}$ and $\tau|_{\mathfrak{g}^\alpha} = \pm\theta|_{\mathfrak{g}^\alpha}$, with $+$ if α is compact and -1 if α non-compact, it follows that θ_ε also commutes with τ .

Lemma II.3. Keep the notation of Definition II.2.

- (i) If ε is a signature of Δ , then the prescription

$$\sigma_\varepsilon(X) := \begin{cases} X & \text{for } X \in \mathfrak{z}_\mathfrak{g}(\mathfrak{a}), \\ \varepsilon(\alpha)X & \text{for } X \in \mathfrak{g}^\alpha, \alpha \in \Delta \end{cases}$$

defines an involutive automorphism of \mathfrak{g} . The involution σ_ε commutes with both τ and θ .

- (ii) Let $\Pi := \{\alpha_1, \dots, \alpha_n\}$ be a basis of Δ . Then for any collection $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$ one can define a signature ε of Δ by setting

$$\varepsilon(\pm \sum_{i=1}^n n_i \alpha_i) := \prod_{i=1}^n \varepsilon_i^{n_i} \quad \text{for} \quad \sum_{i=1}^n n_i \alpha_i \in \Delta.$$

- (iii) Let the notation be as in (ii). Then $\varepsilon \mapsto (\varepsilon(\alpha_i))_{i=1}^n$ defines a bijection between the set of signatures of Δ and $\{-1, 1\}^n$.

Proof. (i) This follows by the Oshima-Sekiguchi construction because $\sigma_\varepsilon = \tau_\varepsilon\theta$. (ii) is clear and (iii) follows from (ii). \blacksquare

In the sequel we identify signatures with elements in $\{-1, 1\}^n$.

Lemma II.4. Let ε be a signature of Δ . Then $\sigma_\varepsilon(\Omega) = \Omega$.

Proof. Let $X \in \Omega$. By (2.2) there is a $k \in H \cap K$ and $Y = \sum_{j=1}^r t_j X_{-j}$, $-1 < t_j < 1$ such that $\text{Ad}(k) \cdot Y = X$. As σ_ε commutes with τ and θ it follows that $\sigma_\varepsilon(k) \in K \cap H$. Hence $\sigma_\varepsilon(X) = \text{Ad}(\sigma_\varepsilon(k)) \cdot \sum_{j=1}^r \varepsilon(\gamma_j) t_j X_{-j} \in \Omega$. \blacksquare

Recall that there is basis $\Pi \subseteq \Delta^+$ having the form

$$\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$$

with α_0 long and non-compact and α_i , $1 \leq i \leq n$ compact. Thus every non-compact negative root $\gamma \in \Delta_n^-$ can be written as $\gamma = -\alpha_0 - \sum_{i=1}^n m_i \alpha_i$, $m_i \in \mathbb{N}_0$. By our choice of Δ_k^+ we have $\alpha_0 = \gamma_1$.

Theorem II.5. *For each $\gamma \in \Delta_n^-$ let $p_\gamma: \mathfrak{n}_n^- \rightarrow \mathfrak{g}^\gamma$ be the orthogonal projection. Then*

$$X \in \Omega \Rightarrow p_\gamma(X) \in \Omega.$$

Proof. Let $X = \sum_{\gamma \in \Delta_n^-} X_\gamma \in \Omega$ with $X_\gamma \in \mathfrak{g}^\gamma$, $\gamma \in \Delta_n^-$. We have to show that $X_\gamma \in \Omega$. Recall that there are at most two root length in Δ (cf. Proposition I.1(i)).

Case 1: γ is a long root.

By Proposition I.1(ii) there exists an element $h \in N_{\text{Inn}(\mathfrak{h} \cap \mathfrak{k})}(\mathfrak{a})$ such that $h.\gamma = -\alpha_0$. Thus we may assume that $\gamma = -\alpha_0 = -\gamma_1$. Let $H := \sum_{j=2}^r H_j$. By Remark II.1 we have

$$X_1 := \lim_{t \rightarrow +\infty} e^{t \text{ad } H}.X \in \Omega.$$

If we express $X_1 = \sum_{\beta \in \Delta_n^-} X_\beta$ as a sum of root vectors, then Proposition I.2 implies that $\beta|_c = -\gamma_1$ or $\beta = \frac{1}{2}\gamma_1 - \delta$ with $\delta(X_0^b) = \frac{1}{2}$. Since $X_0^b \in C_{\max}$ (cf. Lemma I.4), we now get

$$X_\gamma = \lim_{t \rightarrow +\infty} e^{t \text{ad } X_0^b}.X_1 \in \Omega.$$

Case 2: γ is a short root.

By Proposition I.1(iii) we may assume that $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$ and by Lemma I.3 we may suppose $\alpha_0 = \gamma_1$, $\alpha_j = \frac{1}{2}(\gamma_{j+1} - \gamma_j)$ for $1 \leq j \leq r-1$. Write

$$X = \sum_{m_i \geq 0} X_{m_1, \dots, m_n},$$

where $X_{m_1, \dots, m_n} \in \mathfrak{g}^{-(\alpha_0 + \sum_{i=1}^n m_i \alpha_i)}$. Then we have to show that $X_{1,0,\dots,0} \in \Omega$. Set

$$X_{\text{ev}} := \sum_{m_n \equiv 0(2)} X_{m_1, \dots, m_n} \quad \text{and} \quad X_{\text{odd}} := \sum_{m_n \equiv 1(2)} X_{m_1, \dots, m_n}.$$

Then $X = X_{\text{ev}} + X_{\text{odd}}$ and we claim that $X_{\text{ev}}, X_{\text{odd}} \in \Omega$. Let $\varepsilon = (1, 1, 1, \dots, -1)$. Then by Lemma II.4 we get:

$$\sigma_\varepsilon(X) = \sigma_\varepsilon(X_{\text{ev}} + X_{\text{odd}}) = X_{\text{ev}} - X_{\text{odd}} \in \Omega.$$

Since Ω is balanced and convex we moreover have

$$X_{\text{ev}} = \frac{1}{2}(X + \sigma_\varepsilon(X)) \in \Omega \quad \text{and} \quad X_{\text{odd}} = \frac{1}{2}(X - \sigma_\varepsilon(X)) \in \Omega.$$

By repeating this argument we thus my assume that

$$X = \sum_{\substack{m_1 \equiv 1(2) \\ m_j \equiv 0(2), j > 1}} X_{m_1, \dots, m_n}.$$

Now we apply the contraction semigroup generated by $H = \sum_{j=3}^r H_j \in C_{\max}$ and obtain

$$X_1 := \lim_{t \rightarrow +\infty} e^{t \text{ad } H}.X \in \Omega.$$

Thus we may assume $X = X_1$ and $X = \sum_{\beta \in \Delta_n^-} X_\beta$ with $-\beta = \gamma, \gamma_1, \gamma_2, \frac{1}{2}(\gamma_1 + \gamma_2) + \beta, -\frac{1}{2}\gamma_1 + \sigma_1, -\frac{1}{2}\gamma_2 + \sigma_2$ and $\beta, \sigma_1, \sigma_2 \in \mathfrak{b}^*$, $\sigma_1(X_0) = \sigma_2(X_0) = \frac{1}{2}$ (cf. Proposition I.2). Write $\beta = -\gamma_1 - \sum_{j=1}^r m_j \alpha_j$. The cases $\beta = \gamma_1$ and $\beta = \gamma_2$ are excluded, since we have $m_1 = 0$, resp. $m_1 = 2$, contradicting $m_1 \equiv 1(2)$. Applying to X the contraction semigroup generated by $X_0^b \in C_{\max}$ excludes the case $\beta = -\frac{1}{2}\gamma_1 + \sigma_1$ and $\beta = -\frac{1}{2}\gamma_2 + \sigma_2$. Let now $Y \in \mathfrak{b}$ such that $\delta_j(Y) > 0$, $1 \leq j \leq l$, and $\beta_j(Y) > 0$, $1 \leq j \leq m$ (cf. Lemma I.3). Then $\Delta_n^+ \subseteq \mathbb{N}_0[\Pi]$ shows that $Y \in C_{\max}$. But then

$$X_\gamma = \lim_{t \rightarrow +\infty} e^{t \text{ad } Y}.X \in \Omega,$$

completing the proof Case 2 and hence of the theorem. ■

Subdomains of rank one

For $\alpha \in \Delta^+$ we set

$$\mathfrak{g}(\alpha) := (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}])'$$

and $\tau(\alpha) := \tau|_{\mathfrak{g}(\alpha)}$. Then $(\mathfrak{g}(\alpha), \tau(\alpha))$ is a symmetric subalgebra of (\mathfrak{g}, τ) of real rank one, that is $\mathfrak{a}(\alpha) := \mathfrak{a} \cap \mathfrak{g}(\alpha)$ is one dimensional. Further we set $\mathfrak{h}(\alpha) := \mathfrak{h} \cap \mathfrak{g}(\alpha)$ etc. We denote by $G(\alpha)$, $A(\alpha)$ etc. the analytic subgroups of G corresponding to $\mathfrak{g}(\alpha)$, $\mathfrak{a}(\alpha)$ etc. Let $H(\alpha) = G(\tau)^{\tau(\alpha)} = G(\alpha) \cap H$.

Assume that $\alpha \in \Delta_n^+$. Then $(\mathfrak{g}(\alpha), \tau(\alpha))$ is NCC and $\mathfrak{n}(\alpha) = \mathfrak{n}_n^+(\alpha) = \mathfrak{g}^\alpha$. Let $\Omega(\alpha) \cong H(\alpha)/(K(\alpha) \cap H(\alpha))$ be the real bounded symmetric domain in $\overline{\mathfrak{n}}(\alpha) = \mathfrak{n}_n^-(\alpha)$.

Lemma II.6. *Let $\alpha \in \Delta_n^+$ and $s_\alpha \in G(\alpha)$ be a representative of the one element big Weyl group $N_{G(\alpha)}(\mathfrak{a}(\alpha))/Z_{G(\alpha)}(\mathfrak{a}(\alpha))$ of $\mathfrak{g}(\alpha)$. Then*

$$(\overline{N}(\alpha) \cap H(\alpha)A(\alpha)N(\alpha)) \dot{\cup} (\overline{N}(\alpha) \cap H(\alpha)s_\alpha A(\alpha)N(\alpha))$$

is open and dense in $\overline{N}(\alpha)$.

Proof. This follows by Matsukis Theorem (cf. [Ma79, Theorem 3]), if we can show that $M(\alpha) := Z_{K(\alpha)}(\mathfrak{a}(\alpha)) \subseteq H(\alpha)$ because $s_\alpha M(\alpha) = M(\alpha)s_\alpha$. Let $F = \exp(i\mathfrak{a}(\alpha)) \cap G(\alpha)$. Then one has $M(\alpha) = FZ_{H(\alpha)_o}(\mathfrak{a}(\alpha))$ by [NÓ99, Lemma 5.7]. But if $f \in F$ then $\tau(\alpha)(f) = f^{-1} = f$, by the same lemma. Hence $F \subseteq H(\alpha)$, which implies that $M(\alpha) \subseteq H(\alpha)$. ■

Theorem II.7. *Let $\alpha \in \Delta_n^+$. Then $\Omega \cap \overline{\mathfrak{n}}(\alpha) = \Omega(\alpha)$.*

Proof. " \supseteq ": This is clear.

" \subseteq ": Note that $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ is open and convex in $\overline{\mathfrak{n}}(\alpha)$. We have

$$(2.3) \quad \exp(\Omega) \cap (H(\alpha)s_\alpha A(\alpha)N(\alpha)) = \emptyset,$$

since $\exp(\Omega) \subseteq HAN$ and $HAN \cap Hs_\alpha AN = \emptyset$ by Matsukis Theorem. In view of (2.3), Lemma II.7 implies that there exists an open dense subset Ω_α of $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ such that $\Omega_\alpha \subseteq \Omega(\alpha)$. Now the assertion follows from the fact that both $\Omega(\alpha)$ and $\Omega \cap \overline{\mathfrak{n}}(\alpha)$ are open and convex. ■

III. The product formula for the c -function

Recall the HAN -decomposition in G from Section I. For each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and g in G we set

$$a_H(g)^\lambda := \begin{cases} 0 & \text{if } g \notin HAN, \\ e^{\lambda(\log g)} & \text{if } g = han \in HAN. \end{cases}$$

For a locally compact group G we write μ_G for a left Haar measure on G .

Definition III.1. (The c -functions) For each $\alpha \in \mathfrak{a}^*$ let $m_\alpha := \dim \mathfrak{g}^\alpha$ and put $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we now set

$$c(\lambda) := \int_{\overline{N}} a_H(\overline{n})^{-(\lambda+\rho)} d\mu_{\overline{N}}(\overline{n}) = \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(\lambda+\rho)} d\mu_{\overline{N}}(\overline{n}),$$

$$c_\Omega(\lambda) := \int_{N_n^-} a_H(\bar{n})^{-(\lambda+\rho)} d\mu_{N_n^-}(\bar{n}) = \int_\Omega a_H(\bar{n})^{-(\lambda+\rho)} d\mu_{N_n^-}(\bar{n}),$$

and

$$c_0(\lambda) := \int_{N_k^-} a_H(\bar{n})^{-(\lambda+\rho)} d\mu_{N_k^-}(\bar{n})$$

whenever the defining integral exist. We write \mathcal{E} , \mathcal{E}_Ω and \mathcal{E}_0 for the domain of definition of c , c_Ω and c_0 , respectively. We call c the *c-function of the non-compactly causal symmetric space G/H* and c_Ω the *c-function of the real bounded symmetric domain Ω* , while c_0 is the usual *c-function of the non-compact Riemannian symmetric space $G(0)/K(0)$* . ■

Remark III.2. (a) The choice of the particular analytic realization G/H of (\mathfrak{g}, τ) as a symmetric space is immaterial for the definition of the *c*-function.

(b) We have $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\Omega$ and for all $\lambda \in \mathcal{E}$ one has the splitting

$$c(\lambda) = c_0(\lambda)c_\Omega(\lambda)$$

(cf. [FHÓ94, Lemma 9.2]).

(c) The *c*-functions can be written as Laplace transforms (cf. [KNÓ98]). Let us explain this for the *c*-function c . For c_0 and c_Ω one has analogous statements.

There exists a positive Radon measure μ on \mathfrak{a} such that

$$(\forall \lambda \in \mathcal{E}) \quad c(\lambda) = \mathcal{L}_\mu(\lambda) := \int_{\mathfrak{a}} e^{\lambda(X)} d\mu(X),$$

i.e., c is the Laplace transform of μ . In particular we see that the domain of definition \mathcal{E} is a tube domain over a convex set, i.e., one has

$$\mathcal{E} = i\mathfrak{a}^* + \mathcal{E}_{\mathbb{R}}$$

with $\mathcal{E}_{\mathbb{R}} \subseteq \mathfrak{a}^*$ a convex subset of \mathfrak{a}^* . One knows that $\text{int } \mathcal{E}$ is non-empty. Moreover, the fact that c is a Laplace transform implies that c is holomorphic on $\text{int } \mathcal{E}$ and that c has no holomorphic extension to a connected open tube domain strictly larger than $\text{int } \mathcal{E}$. ■

Now we are going to prove the product formula for the *c*-function c_Ω . Our strategy is a modified Gindikin-Karpelevic approach as presented in [GaVa88, p. 175–177] or [Hel84, Ch. IV].

For a positive system $R \subseteq \Delta$ we set $\bar{n}_R := \bigoplus_{\alpha \in -(\Delta^+ \cap R)} \mathfrak{g}^\alpha$ and write \bar{N}_R for the corresponding analytic subgroup of G . We define an auxiliary *c*-function by

$$c_R(\lambda) := \int_{\bar{N}_R} a_H(\bar{n})^{-(\lambda+\rho)} d\mu_{\bar{N}_R}(\bar{n})$$

whenever the integral exists.

For a single root $\alpha \in \Delta^+$ we set $\rho_\alpha := \frac{1}{2}m_\alpha\alpha$ and write

$$c_\alpha(\lambda) := \int_{\bar{N}(\alpha)} a_{H(\alpha)}(\bar{n})^{-(\lambda+\rho_\alpha)} d\mu_{\bar{N}(\alpha)}(\bar{n}).$$

We denote by $\mathcal{E}_\alpha \subseteq \mathfrak{a}_C^*$ the domain of definition of c_α .

Proposition III.3. *For any positive system $R \subseteq \Delta$ we have that*

$$c_R(\lambda) = \prod_{\alpha \in (R \cap \Delta^+)} c_\alpha(\lambda)$$

and $c_R(\lambda)$ is defined if and only if $\lambda \in \bigcap_{\alpha \in (R \cap \Delta^+)} \mathcal{E}_\alpha$.

Proof. We proceed by induction on $|R \cap \Delta^+|$. If $R \cap \Delta^+ = \emptyset$, then the assertion is clear.

Assume that $R \cap \Delta^+ \neq \emptyset$. Then we find an element $\beta \in R \cap \Delta^+$ which is simple in R . Set $Q := s_\beta \cdot R$. Then $Q = R \setminus \{\beta\} \cup \{-\beta\}$ since Δ is reduced (cf. Proposition I.1(i)). Thus we have $(Q \cap \Delta^+) \dot{\cup} \{\beta\} = R \cap \Delta^+$. We now have to distinguish to cases.

Case 1: β is compact.

In this case, the HAN -decomposition of $G(\beta)$ coincides with the Iwasawa decomposition, i.e. $G(\beta) = K(\beta)A(\beta)N(\beta)$. Thus $c_R(\lambda) = c_\beta(\lambda)c_Q(\lambda)$ follow as in [GaVa88, Prop. 4.7.6].

Case 2: β is non-compact.

Set $\overline{N}_Q^k := \overline{N}_Q \cap N_k^-$, $\overline{N}_Q^n := \overline{N}_Q \cap N_n^-$ and note that $\overline{N}_Q = \overline{N}_Q^n \rtimes \overline{N}_Q^k$. Since $\overline{N}_R = \overline{N}(\beta)\overline{N}_Q$ we thus get

$$c_R(\lambda) = \int_{\overline{N}(\beta)} \int_{\overline{N}_Q^n} \int_{\overline{N}_Q^k} a_H(\overline{n}_\beta \overline{n}_n \overline{n}_k)^{-(\lambda+\rho)} d\mu_{\overline{N}(\beta)}(\overline{n}_\beta) d\mu_{\overline{N}_Q^n}(\overline{n}_n) d\mu_{\overline{N}_Q^k}(\overline{n}_k).$$

If $\overline{n}_\beta \overline{n}_n \overline{n}_k \in \overline{N} \cap HAN$, then (1.1) implies that $\overline{n}_\beta \overline{n}_n \in \exp(\Omega)$. Since \mathfrak{n}_n^- is abelian, Theorem II.5 therefore implies that $\overline{n}_\beta \in \exp(\Omega)$ and so $\overline{n}_\beta \in \exp(\Omega(\beta))$ by Theorem II.7. Therefore we can write $\overline{n}_\beta = h_\beta a_\beta n_\beta$ with $h_\beta \in H(\beta)$, $a_\beta \in A(\beta)$ and $n_\beta \in N(\beta)$. Now one can proceed as in [GaVa88, p. 175–177] and one gets $c_R(\lambda) = c_\beta(\lambda)c_Q(\lambda)$. ■

Remark III.4. If we choose $R = -\Delta_n^+ \cup \Delta_k^+$ (this is a positive system since Δ_n^+ is \mathcal{W} -invariant), then we have $c_0 = c_R$ and Proposition III.3 results in the Gindikin-Karpelevic product formula

$$c_0(\lambda) = \prod_{\alpha \in \Delta_k^+} c_\alpha(\lambda)$$

of the c -function c_0 on $G(0)/K(0)$ (cf. [GaVa88, Th. 4.7.5] or [Hel84, Ch. IV, Th. 6.13, 6.14]). ■

Theorem III.5. (The product formula for c_Ω) *For the c -function c_Ω of the real bounded symmetric domain Ω one has*

$$\mathcal{E}_\Omega = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*: (\forall \alpha \in \Delta_n^+) \operatorname{Re} \lambda(H_\alpha) < 2 - m_\alpha\}$$

and

$$c_\Omega(\lambda) = \kappa \prod_{\alpha \in \Delta_n^+} B\left(\frac{m_\alpha}{2}, -\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right)$$

where B denotes the Beta function and κ is a positive constant only depending on (\mathfrak{g}, τ) .

Proof. Set $\mathcal{E}'_\Omega := \bigcap_{\alpha \in \Delta_n^+} \mathcal{E}_\alpha$. We want to apply Proposition III.3 to $R = \Delta^+$. In view of Remark III.2(b) and Remark III.4, we thus get

$$(3.1) \quad (\forall \lambda \in \mathcal{E} \cap \mathcal{E}'_\Omega) \quad c_\Omega(\lambda) = \prod_{\alpha \in \Delta_n^+} c_\alpha(\lambda) = \prod_{\alpha \in \Delta_n^+} c_{\Omega(\alpha)}(\lambda).$$

By [FHÓ94, (10.3)] one has

$$(3.2) \quad c_{\Omega(\alpha)}(\lambda) = 2^{m_\alpha-1} B\left(\frac{m_\alpha}{2}, -\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1\right)$$

and

$$(3.3) \quad \mathcal{E}_{\Omega(\alpha)} = \{\lambda \in \mathfrak{a}_\mathbb{C}^*: \operatorname{Re} \lambda(H_\alpha) < 2 - m_\alpha\}.$$

It follows from (3.3) that

$$(3.4) \quad \mathcal{E}'_\Omega = \{\lambda \in \mathfrak{a}_\mathbb{C}^*: (\forall \alpha \in \Delta_n^+) \operatorname{Re} \lambda(H_\alpha) < 2 - m_\alpha\}.$$

Besides $\mathcal{E}_\Omega = \mathcal{E}'_\Omega$ all assertions of the theorem now follow from (3.1)-(3.4). Finally, $\mathcal{E}_\Omega = \mathcal{E}'_\Omega$ follows from the fact that all c -functions involved are Laplace transforms (cf. Remark III.2(c)). ■

The following simple fact that shows that we can split off all the non-compact roots to get the c_Ω -function before we come to the compact roots.

Lemma III.6. *Let R be a any positive system of roots in Δ . If $R \cap \Delta_n^+ \neq \emptyset$, then $R \cap \Delta_n^+$ contains a root that is simple in R .*

Proof. Let $\{\beta_0, \dots, \beta_n\}$ be the set of simple roots in R . Let $\gamma \in R \cap \Delta_n^+$. Then $\gamma = \sum_{i=0}^n n_i \beta_i$ with $n_i \in \mathbb{N}_0$. Thus $1 = \gamma(X_0) = \sum_{i=0}^n n_i \beta_i(X_0)$ which implies that $\beta_i(X_0) > 0$ for at least one β_i . But then $\beta_i \in \Delta_n^+$. ■

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